

INEQUALITIES FOR GENERALIZED MINORS

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ABSTRACT. It is a classical result that the absolute value of any k -minor of an $r \times s$ real or complex matrix is bounded by the product of its first k singular values. We generalize this statement to the context of real or complex simple Jordan pairs with generalized minors given by Jordan algebra determinants.

INTRODUCTION

The goal of this paper is a generalization of the following statement.

Let A be a real or complex $r \times s$ matrix, $r \leq s$, with singular values $\sigma_1 \geq \dots \geq \sigma_r \geq 0$. Then for $1 \leq k \leq r$ the absolute value of any k -minor of A is bounded by the product $\sigma_1 \cdots \sigma_k$.

This estimate can be proved using the Cauchy–Binet formula, see e.g. [2, Theorem 4.1]. We generalize this statement to the context of Jordan theory. Let (V, \overline{V}) be a real or complex simple Jordan pair with positive involution. Then any element $z \in V$ admits a singular value decomposition, and to any tripotent $e \in V$ there is associated a unital Jordan algebra $[e] \subseteq V$ and a Jordan algebra determinant Δ_e .

THEOREM. *Let (V, \overline{V}) be a real or complex simple Jordan pair with positive involution and rank r . Let $e \in V$ be a tripotent of rank k , and $z \in V$ be an element with singular values $\sigma_1 \geq \dots \geq \sigma_r \geq 0$. Then*

$$|\Delta_e(z)| \leq \sigma_1 \cdots \sigma_k.$$

Our proof is analytic and quite elementary. Let us describe the main idea in the classical context with $V = \mathbb{k}^{r \times s}$, $\mathbb{k} = \mathbb{R}$ or \mathbb{C} . Then an element $e \in \mathbb{k}^{r \times s}$ is tripotent if it satisfies the identity $e = ee^*e$, and it turns out that the corresponding Jordan algebra determinant Δ_e is given by

$$\Delta_e(z) = \text{Det}(\mathbf{1}_r + (e - z)e^*) \quad (z \in \mathbb{k}^{r \times s}).$$

Moreover, the set S_k of all rank- k tripotents forms a real smooth compact submanifold of $\mathbb{k}^{r \times s}$. Now the theorem follows from plain analysis of the smooth map $f : S_k \rightarrow \mathbb{R}$ given by $f(e) = |\Delta_e(z)|^2$ with fixed $z \in \mathbb{k}^{r \times s}$.

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We note that choosing the tripotent e appropriately, $\Delta_e(z)$ coincides with a given k -minor of z , see Example 1.3 for details. Therefore, the theorem indeed generalizes the classical statement.

In the first section, we discuss generalized minors in the context of simple Jordan pairs over arbitrary fields of characteristic $\neq 2$ in some detail. We note that even though the basic definition of generalized minors (resp. the corresponding Jordan algebra determinants) is widely known, our results seem to be new. In Section 2 we specialize to the case of real or complex simple Jordan pairs, prove our main theorem, and provide an application involving some representation theory on the space of polynomials on V , see Corollary 2.4.

1. GENERALIZED MINORS

In this section we consider Jordan pairs without fixing an involution. Let (V^+, V^-) be a finite dimensional simple Jordan pair over a field \mathbb{k} of characteristic $\neq 2$, and with quadratic maps $Q^\pm : V^\pm \rightarrow \text{Hom}(V^\mp, V^\pm)$. As usual, we omit the signs and simply write $Q_z := Q^\pm(z)$ for $z \in V^\pm$. Moreover, for $x, z \in V^\pm$ and $y \in V^\mp$ the Jordan triple product $\{x, y, z\}$ and the operators $D_{x,y}$ and $Q_{x,z}$ are defined by polarization of Q_x ,

$$\{x, y, z\} := D_{x,y}z := Q_{x,z}y := Q_{x+z}y - Q_xy - Q_zy.$$

The results of this section are independent of the choice of \mathbb{k} . We refer to [5, 6] for a detailed introduction to Jordan pairs. We briefly recall some basic notions necessary for our purposes.

An element $\mathbf{e} = (e_+, e_-)$ in (V^+, V^-) is *idempotent*, if $e_+ = Q_{e_+}e_-$ and $e_- = Q_{e_-}e_+$. The corresponding *Peirce decomposition* is given by

$$V^\pm = V_2^\pm(\mathbf{e}) \oplus V_1^\pm(\mathbf{e}) \oplus V_0^\pm(\mathbf{e}) \quad \text{with} \quad V_\nu^\pm(\mathbf{e}) := \{z \in V^\pm \mid D_{e_+, e_\mp} z = \nu z\}.$$

For the following fix $\sigma \in \{+, -\}$. The Peirce 2-space $V_2^\sigma(\mathbf{e})$ coincides with the *principal inner ideal* $[e_\sigma] := Q_{e_\sigma}V^{-\sigma}$ generated by e_σ . Moreover, $[e_\sigma]$ forms a unital Jordan algebra with product $x \circ y := \frac{1}{2}\{x, e_{-\sigma}, y\}$ and unit element e_σ , see also Remark 1.2. By definition, the *Jordan algebra determinant* $\Delta_\mathbf{e}^\sigma$ of $[e_\sigma]$ is the exact denominator of the rational map $z \mapsto z^{-1}$, normalized to $\Delta_\mathbf{e}^\sigma(e_\sigma) = 1$. As usual, we expand $\Delta_\mathbf{e}^\sigma$ to a polynomial on all of V^σ by firstly projecting onto $[e_\sigma]$ (along $V_1^\sigma(\mathbf{e}) \oplus V_0^\sigma(\mathbf{e})$) and then evaluating $\Delta_\mathbf{e}^\sigma$. By abuse of notation, this expansion is also denoted by $\Delta_\mathbf{e}^\sigma$ and called the *generalized minor* associated to \mathbf{e} .

There is also a determinant attached to the Jordan pair (V^+, V^-) , which we describe next. A pair $(x, y) \in V^\sigma \times V^{-\sigma}$ is *quasi-invertible*, if the *Bergman operator* $B_{x,y} := \text{Id} - D_{x,y} + Q_xQ_y$ is invertible. In this case,

$$x^y := B_{x,y}^{-1}(x - Q_x y)$$

is the *quasi-inverse* of (x, y) . The exact denominator $\Delta : V^+ \times V^- \rightarrow \mathbb{k}$ of the rational map $(x, y) \mapsto x^y$, normalized to $\Delta(0, 0) = 1$, is the *Jordan pair determinant* (also often called the *generic norm*, see [5, § 16.9]).

There is a simple relation between Jordan algebra determinants and the Jordan pair determinant, which we already noted in [9] for the special case of complex simple Jordan pairs. To the best of our knowledge this relation has not been stated elsewhere.

PROPOSITION 1.1. *Let $\mathbf{e} = (e^+, e^-)$ be an idempotent of (V^+, V^-) . Then*

$$(1.1) \quad \Delta_{\mathbf{e}}^+(x) = \Delta(e_+ - x, e_-), \quad \Delta_{\mathbf{e}}^-(y) = \Delta(e_+, e_- - y)$$

for all $x \in V^+$, $y \in V^-$.

Proof. Due to the duality of the Jordan pairs (V^+, V^-) and (V^-, V^+) , it suffices to prove the first formula in (1.1). If the Jordan pair is the one associated to a unital Jordan algebra J , i.e., $(V^+, V^-) = (J, J)$, and if e_+ is the unit element of J , then (1.1) is an immediate consequence of [5, § 16.3(ii)]. We reduce the general case to this Jordan algebra case. Let $x = x_2 + x_1 + x_0$ be the decomposition of x according to the Peirce decomposition with respect to \mathbf{e} . By definition, $\Delta_{\mathbf{e}}^+(x) = \Delta_{\mathbf{e}}^+(x_2)$. On the other hand, due to [5, § 3.5] the relation $\Delta(u, Q_v w) = \Delta(w, Q_v u)$ holds for all $u, w \in V^+$, $v \in V^-$, so it follows that $\Delta(e_+ - x, e_-) = \Delta(e_+ - x_2, e_-)$ since $e_- = Q_{e_-} e_+$ and $Q_{e_-} x = Q_{e_-} x_2$. Therefore it suffices to assume $x = x_2$. We may consider $([e_+], [e_-])$ as a subpair of (V^+, V^-) . Then its Jordan pair determinant coincides with the restriction of Δ to $[e_+] \times [e_-]$. Moreover, due to [5, § 1.11] we may identify $([e_+], [e_-])$ with the Jordan pair (J, J) with $J = [e_+]$ via the Jordan algebra isomorphism $Q_{e_-} : [e_+] \rightarrow [e_-]$. Now we are in the Jordan algebra case, and (1.1) follows from [5, § 16.3(ii)]. \square

REMARK 1.2. We note that the Jordan algebra structure on $[e_\sigma] = V_2^\sigma(\mathbf{e})$ is independent of $e_{-\sigma}$, since for $x = Q_{e_\sigma} u$ in $[e_\sigma]$, the fundamental formula for the quadratic map yields

$$x^2 = Q_x e_{-\sigma} = Q_{Q_{e_\sigma} u} e_{-\sigma} = Q_{e_\sigma} Q_u Q_{e_\sigma} e_{-\sigma} = Q_{e_\sigma} Q_u e_\sigma,$$

and polarization of x^2 also shows that $x \circ y$ is independent of $e_{-\sigma}$. It follows that the Jordan algebra determinant $\Delta_{\mathbf{e}}^\sigma : [e_\sigma] \rightarrow \mathbb{k}$ does not depend on $e_{-\sigma}$. However, its expansion to V^σ depends on $e_{-\sigma}$ since the Peirce spaces $V_1^\sigma(\mathbf{e})$ and $V_0^\sigma(\mathbf{e})$ are dependent on $e_{-\sigma}$.

EXAMPLE 1.3. Consider the simple Jordan pair $(\mathbb{k}^{r \times s}, \mathbb{k}^{s \times r})$ with $r \leq s$ and quadratic maps given by $Q_x y = xyx$. Then, idempotents are pairs of matrices (e_+, e_-) satisfying $e_+ e_- e_+ = e_-$ and $e_- e_+ e_- = e_+$, and the Jordan pair determinant is given by $\Delta(x, y) = \text{Det}(\mathbf{1}_r - xy)$. For $1 \leq k \leq r$ and tuples $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_k)$ with $1 \leq i_1 < \dots < i_k \leq r$

and $1 \leq j_1 < \dots < j_k \leq s$ let e_+ be the $r \times s$ -matrix defined by

$$(1.2) \quad (e_+)_{ij} := \begin{cases} 1 & \text{if } (i, j) = (i_\ell, j_\ell) \text{ for some } 1 \leq \ell \leq k, \\ 0 & \text{else,} \end{cases}$$

and set $e_- := e_+^\top$, the transpose matrix of e_+ . Then, it is straightforward to show that (e_+, e_-) is an idempotent, and the generalized minor

$$\Delta_{\mathbf{e}}^+(z) = \text{Det}(\mathbf{1}_r - (e_+ - z)e_-) \quad (z \in \mathbb{k}^{r \times s})$$

coincides with the usual minor corresponding to rows and columns given by I and J , respectively.

PROPOSITION 1.4. *Let $\mathbf{e}, \mathbf{c} \in (V^+, V^-)$ be idempotents. If $[e_+] = [c_+]$, then*

- (i) $\Delta_{\mathbf{e}}^+(x) = \Delta_{\mathbf{e}}^+(c_+) \cdot \Delta_{\mathbf{c}}^+(x)$ for all $x \in [e_+]$,
- (ii) $\Delta_{\mathbf{e}}^-(y) = \Delta_{\mathbf{e}}^-(c_-) \cdot \Delta_{\mathbf{c}}^-(y)$ for all $y \in V^-$,
- (iii) $\Delta_{\mathbf{c}}^+(e_+) \cdot \Delta_{\mathbf{c}}^-(e_-) = 1$.

If $[e_-] = [c_-]$, the same formulas hold when $+$ and $-$ are interchanged.

Proof. The first formula is well-known from the theory of mutations of Jordan algebras, see e.g. [1, V. § 3]. The second formula needs different arguments, since $[e_-]$ might differ from $[c_-]$. We claim that $(e_+, e_- - c_-)$ is quasi-invertible with quasi-inverse $e_+^{e_- - c_-} = c_+$. In this case, (ii) follows from standard identities of the Jordan pair determinant [5, § 16.11],

$$\begin{aligned} \Delta_{\mathbf{e}}^-(y) &= \Delta(e_+, e_- - y) \\ &= \Delta(e_+, e_- - c_- + c_- - y) \\ &= \Delta(e_+, e_- - c_-) \Delta(e_+^{e_- - c_-}, c_- - y) \\ &= \Delta_{\mathbf{e}}^-(c_-) \Delta(c_+, c_- - y) \\ &= \Delta_{\mathbf{e}}^-(c_-) \Delta_{\mathbf{c}}^-(y). \end{aligned}$$

In order to show quasi-invertibility of $(e_+, e_- - c_-)$, consider the decomposition $c_- = c_2 \oplus c_1 \oplus c_0$ of c_- according to the Peirce decomposition of V^- with respect to \mathbf{e} . Since \mathbf{c} is an idempotent, the Peirce rules [5, § 5.4] yield the following relations:

$$Q_{c_+} c_2 = c_+, \quad Q_{c_2} c_+ \oplus \{c_2, c_+, c_1\} \oplus Q_{c_1} c_+ = c_2 \oplus c_1 \oplus c_0.$$

Comparing the components of the Peirce spaces in the second identity, we conclude that the pair (c_+, c_2) is also idempotent with $[c_+] = [e_+]$ and $[c_2] = [e_-]$. By assumption, c_+ is invertible in the Jordan algebra $[e_+]$. Therefore, c_2 is invertible in the Jordan algebra $[e_-]$, and it follows that $(e_- - c_2, e_+)$ is quasi-invertible with quasi-inverses

$$(e_- - c_2)^{e_+} = c_2^{-1} - e_-,$$

where c_2^{-1} is the inverse of c_2 in $[e_-]$, see [5, § 3.1]. Now recall that Jordan algebra inverses satisfy $a^{-1} = P_a^{-1}a$, where P_a is the quadratic operator corresponding to $a \in [e_-]$. Here, $P_a = Q_a Q_{e_+}|_{[e_-]}$, so we obtain

$$c_2^{-1} = (Q_{c_2} Q_{e_+}|_{[e_-]})^{-1} c_2 = Q_{e_-} (Q_{c_2}|_{[c_2]})^{-1} c_2 = Q_{e_-} c_+.$$

Due to the symmetry formula [5, § 3.3] for quasi-inverses it follows that

$$e_+^{e_- - c_2} = e_+ + Q_{e_+}(e_- - c_2)^{e_+} = e_+ + Q_{e_+}(Q_{e_-} c_+ - e_-) = c_+.$$

Finally, the shifting formula [5, § 3.5] yields that $(e_+, e_- - c_-)$ is quasi-invertible if and only if $(e_+, e_- - c_2)$ is quasi-invertible, and both have the same quasi-inverse. It remains to show (iii). Since $\Delta(Q_u v, w) = \Delta(Q_u w, v)$ for all $u \in V^+$, $v, w \in V^-$, Proposition 1.1 yields the relation $\Delta_{\mathbf{e}}^-(c_-) = \Delta_{\mathbf{e}}^+(Q_{e_+} c_-)$. Now applying (i), we obtain

$$(1.3) \quad \Delta_{\mathbf{e}}^-(c_-) = \Delta_{\mathbf{e}}^+(c_+) \cdot \Delta_{\mathbf{c}}^+(Q_{e_+} c_-).$$

Since $Q_{e_+} c_- = e_+^2$ in the Jordan algebra $[c_+]$, the second term on the right hand side becomes $\Delta_{\mathbf{c}}^+(e_+)^2$. Moreover, setting $x = e_+$ in (i) yields $\Delta_{\mathbf{c}}^+(e_+) = \Delta_{\mathbf{e}}^+(c_+)^{-1}$, so (1.3) implies (iii). By duality of the Jordan pairs (V^+, V^-) and (V^-, V^+) the same statement holds when $+$ and $-$ are interchanged. \square

REMARK 1.5. We note that Proposition 1.4(i) does not necessarily hold for all $x \in V^\sigma$, as the following calculation illustrates. As in Example 1.3 consider the Jordan pair $(\mathbb{k}^{r \times s}, \mathbb{k}^{r \times s})$. Let $\mathbf{e} = (e_+, e_-)$ be defined by

$$e_+ = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} A^{-1} & B \\ C & CAB \end{pmatrix},$$

where $A \in \mathbb{k}^{k \times k}$ is invertible, and $B \in \mathbb{k}^{k \times (s-k)}$, $C \in \mathbb{k}^{(r-k) \times k}$ are arbitrary. Any such pair of matrices is an idempotent of $(\mathbb{k}^{r \times s}, \mathbb{k}^{r \times s})$, and yields the same principal inner ideal in $\mathbb{k}^{r \times s}$,

$$[e_+] = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a \in \mathbb{k}^{k \times k} \right\}.$$

The generalized minors are given by

$$\Delta_{\mathbf{e}}^+(x) = \text{Det}(\mathbf{1}_r - (e_+ - x)e_-) = \text{Det} \begin{pmatrix} aA^{-1} + bC & -AB + aB + bCAB \\ cA^{-1} + dC & 1 + cB + dCAB \end{pmatrix}$$

where $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{k}^{r \times s}$ with $a \in \mathbb{k}^{k \times k}$, and

$$\Delta_{\mathbf{e}}^-(y) = \text{Det}(\mathbf{1}_r - e_+(e_- - y)) = \text{Det} \begin{pmatrix} A\alpha & -A(B - \beta) \\ 0 & \mathbf{1}_{r-k} \end{pmatrix} = \text{Det } A \cdot \text{Det } \alpha$$

where $y = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{k}^{s \times r}$ with $\alpha \in \mathbb{k}^{k \times k}$. We see that $\Delta_{\mathbf{e}}^+(x)$ simplifies to $\text{Det } A^{-1} \cdot \text{Det } a$ only if $x \in [e_+]$ or $B = 0$ and $C = 0$, so Proposition 1.4(i) fails for $x \notin [e_+]$, e.g. in the case where $B \neq 0$ and $\mathbf{c} = ((\frac{1}{0} \ 0)), (\frac{1}{0} \ 0))$.

REMARK 1.6. Strictly speaking, Proposition 1.4(ii) is not needed for the purpose of this paper. However we find it worthwhile to include this formula, since it is a rather suprising identity, in particular considered in contrast to Proposition 1.4(i) with its restricted domain.

2. INEQUALITIES FOR GENERALIZED MINORS

In this section, let (V, \bar{V}) be a simple Jordan pair over $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$ with positive involution $\vartheta : V \rightarrow \bar{V}$. For convenience, we set $\bar{z} := \vartheta(z)$ for $z \in V$. Recall that an element $e \in V$ is a *tripotent*, if $\mathbf{e} := (e, \bar{e})$ is an idempotent. All notions and results of the last section also apply to this idempotent, and since e uniquely determines the idempotent, we simplify the notation and set

$$V_\nu(e) := V_\nu^+(\mathbf{e}), \quad \Delta_e(x) := \Delta_{\mathbf{e}}^+(x),$$

for $\nu = 2, 1, 0$ and $x \in V$.

Two tripotents $e, c \in V$ are *strongly orthogonal*, if $e \in V_0(c)$ (or equivalently $c \in V_0(e)$), and e is called *primitive*, if it cannot be written as a sum of two non-zero strongly orthogonal tripotents. Any tripotent is the sum of primitive tripotents, and the number of summands is called its *rank*. A *frame* is a maximal system (e_1, \dots, e_r) of primitive strongly orthogonal tripotents. Here, r is independent of the choice of the frame, and called the *rank* of the Jordan pair V .

Due to [6, §3.12] any element $z \in V$ admits a singular value decomposition, i.e.,

$$(2.1) \quad z = \sigma_1 e_1 + \dots + \sigma_r e_r$$

where (e_1, \dots, e_r) is a frame of tripotents and $\sigma_1 \geq \dots \geq \sigma_r \geq 0$ are uniquely determined real numbers, called the *singular values* of z .

EXAMPLE 2.1. In the case $V = \mathbb{k}^{r \times s}$ discussed in our previous examples, a positive involution is given by the Hermitian transpose, $z \mapsto z^* \in \mathbb{k}^{s \times r}$. Then, an element $e \in \mathbb{k}^{r \times s}$ is a tripotent if and only if $e = ee^*e$, i.e., if e is a partial isometry. The element constructed in (1.2) is in fact a tripotent. The singular value decomposition (2.1) coincides with the usual one. Indeed, let $z = U_1 \Sigma U_2$ be the usual singular value decomposition with $U_1 \in U_r(\mathbb{k})$, $U_2 \in U_s(\mathbb{k})$ and diagonal matrix $\Sigma \in \mathbb{k}^{r \times s}$ with entries $\sigma_1 \geq \dots \geq \sigma_r \geq 0$. For $1 \leq k \leq r$ set $e_i := U_1 E_i U_2$ where E_i denotes the matrix with 1 at the (i, i) 'th position and 0 elsewhere. Then (e_1, \dots, e_r) is a frame of tripotents, and $z = \sigma_1 e_1 + \dots + \sigma_r e_r$.

THEOREM 2.2. *Let $e \in V$ be a tripotent of rank k , and $z \in V$ be an element with singular values $\sigma_1 \geq \dots \geq \sigma_r \geq 0$. Then*

$$|\Delta_e(z)| \leq \sigma_1 \cdots \sigma_k.$$

Before proving this, we determine the derivative of the Jordan pair determinant Δ . Since (V, \bar{V}) is assumed to be simple, it follows from [5, § 17.3] that Δ is irreducible and satisfies

$$(2.2) \quad \text{Det } B_{x,y} = \Delta(x, y)^p,$$

where p is a structure constant of (V, \bar{V}) , and Det denotes the standard determinant of the Bergman operator $B_{x,y} \in \text{End}(V)$. Recall that

$$(2.3) \quad \tau : V \times \bar{V} \rightarrow \mathbb{k}, (x, y) \mapsto \text{Tr } D_{x,y}$$

is a non-degenerate pairing, called the *trace form* of (V, \bar{V}) .

LEMMA 2.3. *The derivative of the Jordan pair determinant at $(x, y) \in V \times \bar{V}$ along $(u, v) \in V \times \bar{V}$ is given by*

$$(2.4) \quad d_{(u,v)} \Delta(x, y) = -\frac{1}{p} \Delta(x, y) (\tau(u, y^x) + \tau(x^y, v)).$$

Proof. We note that it suffices to prove (2.4) for generic $(x, y) \in V \times \bar{V}$, so we may assume that $\Delta(x, y) \neq 0$, or equivalently that $B_{x,y}$ is invertible. Taking derivatives on both sides of (2.2) yields

$$p \Delta(x, y)^{p-1} d_{(u,v)} \Delta(x, y) = \text{Det } B_{x,y} \text{Tr} (B_{x,y}^{-1} d_{(u,v)} B_{x,y}),$$

and hence

$$d_{(u,v)} \Delta(x, y) = \frac{1}{p} \Delta(x, y) \text{Tr} (B_{x,y}^{-1} d_{(u,v)} B_{x,y}).$$

Since $d_{(u,v)} B_{x,y} = -D_{u,y} + Q_{x,u} Q_y - D_{x,v} + Q_x Q_{y,v}$, it follows that

$$\text{Tr} (B_{x,y}^{-1} d_{(u,v)} B_{x,y}) = -\text{Tr } D_{u,y^x} - \text{Tr } D_{x^y,v} = -\tau(u, y^x) - \tau(x^y, v),$$

where we used the relations $D_{u,y^x} B_{x,y} = D_{u,y} - Q_{x,u} Q_y$ and $B_{x,y} D_{x^y,v} = D_{x,v} - Q_x Q_{y,v}$, see the appendix of [6]. This completes the prove. \square

Proof of Theorem 2.2. Let $S_k \subseteq V$ denote the subset of tripotents of rank k . It is known [6, §§ 5.6, 11.12] that S_k is a compact submanifold, and the tangent space $T_e S_k$ at $e \in S_k$ is given in terms of the Peirce decomposition of V with respect to e in the following way: Recall that the map $x \mapsto x^\# := Q_e \bar{x}$ defines an involution on $V_2(e)$ with eigenspace decomposition $V_2(e) = A(e) \oplus B(e)$ where

$$A(e) := \{x \in V_2(e) \mid x = x^\#\}, \quad B(e) := \{x \in V_2(e) \mid x = -x^\#\}.$$

Then $T_e S_k = B(e) \oplus V_1(e)$. Moreover, recall that the decomposition $V = A(e) \oplus B(e) \oplus V_1(e) \oplus V_0(e)$ is orthogonal with respect to the positive definite inner product $(u, v) := \tau(u, \bar{v})$ on V . For fixed $z \in V$, we determine the maximum value of the map $f : S_k \rightarrow \mathbb{R}$, $f(e) := |\Delta_e(z)|^2$. Due to Lemma 1.1, it is clear that f is smooth, and Lemma 2.3 implies that the derivative of f at $e \in S_k$ along $u = u_2 + u_1 \in T_e S_k$ is given by

$$d_u f(e) = -\frac{2}{p} f(e) \cdot \text{Re} (\tau(u, (\bar{e})^{e-z}) + \tau((e-z)^{\bar{e}}, \bar{u})).$$

The symmetry formula for quasi-inverses [5, §3.3] yields $(\bar{e})^{e-z} = \bar{e} + Q_{\bar{e}}(e-z)^{\bar{e}}$, and since $\tau(x, Q_y z) = \tau(z, Q_y x)$ it follows that

$$\tau(u, \bar{e}^{e-z}) = \tau(u, \bar{e} + Q_{\bar{e}}(e-z)^{\bar{e}}) = \tau(u, \bar{e}) + \tau((e-z)^{\bar{e}}, Q_{\bar{e}}u).$$

We thus obtain

$$d_u f(e) = -\frac{2}{p} f(e) \cdot \operatorname{Re} \tau((e-z)^{\bar{e}}, \bar{u}_1),$$

since $e \in A(e) \perp T_e S_k$ and $Q_{\bar{e}}u + \bar{u} = \bar{u}_1$. Therefore, f attains its maximum value at e only if $\tau((e-z)^{\bar{e}}, \bar{u}_1) = 0$ for all $u_1 \in V_1(e)$, i.e., only if $(e-z)^{\bar{e}} = x_2 + x_0 \in V_2(e) \oplus V_0(e)$. In this case, the shifting formula for quasi-inverses [5, §3.5] yields

$$e - z = (x_2 + x_0)^{-\bar{e}} = x_2^{-\bar{e}} + x_0 \in V_2(e) \oplus V_0(e),$$

so it follows that $z = z_2 + z_0 \in V_2(e) \oplus V_0(e)$. Now, strong orthogonality of $V_2(e)$ and $V_0(e)$ implies that the singular value decomposition of z splits into the corresponding decompositions of z_2 and z_0 , so $\{\sigma_1, \dots, \sigma_r\} = \{\mu_1, \dots, \mu_k\} \cup \{\nu_1, \dots, \nu_{r-k}\}$, where the μ_i (resp. ν_i) are the singular values of z_2 (resp. z_0). We claim that $|\Delta_e(z)| = \prod \mu_i$. If so, then f attains its maximum value if $\mu_i = \sigma_i$ for $i = 1, \dots, k$, and we are finished. To evaluate $|\Delta_e(z)|$ first recall that $\Delta_e(z) = \Delta_e(z_2)$. Let $z_2 = \mu_1 c_1 + \dots + \mu_k c_k$ denote the spectral decomposition of z_2 . We note that this is not necessarily the spectral decomposition of $z_2 \in [e]$ in the sense of Jordan algebras [3, III.1.1], since the sum $c := c_1 + \dots + c_k$ might differ from e . However, since $[c] = [e]$, Proposition 1.4(i) yields $\Delta_e(z_2) = \Delta_c(e)^{-1} \Delta_c(z_2)$. Now, due to Proposition 1.1 and [5, §16.15] we obtain

$$\Delta_c(z_2) = \Delta(c - z_2, \bar{c}) = \prod (1 - (1 - \mu_i) \cdot 1) = \prod \mu_i.$$

Finally, recall that $\Delta(u, \bar{v}) = \overline{\Delta(v, \bar{u})}$ for all $u, v \in V$, where $\bar{\alpha}$ denotes complex conjugation of $\alpha \in \mathbb{k}$. Therefore, Proposition 1.4(iii) yields $|\Delta_c(e)| = 1$, and we conclude that $|\Delta_e(z_2)| = \prod \mu_i$. This completes the proof. \square

We finally give an application of Theorem 2.2 involving some representation theory. Consider a *complex* simple Jordan pair (V, \bar{V}) with involution of rank r , and let $\mathcal{P}(V)$ denote the space of complex polynomial maps on V . Let L denote the identity component of the structure group associated to (V, \bar{V}) , which consists of linear maps $h \in \operatorname{GL}(V)$ satisfying the relation $h\{x, y, z\} = \{hx, h^{-\#}y, hz\}$ where $h^{-\#}$ denotes the inverse of the adjoint of h with respect to the trace form τ defined in (2.3). Then L is a reductive complex Lie group with maximal compact subgroup $K := L \cap U(V)$, where $U(V)$ denotes unitary operators with respect to the inner product $(u, v) = \tau(u, \bar{v})$ on V . The induced action of K on polynomials yields a decomposition of $\mathcal{P}(V)$ into irreducible components. Due to Hua, Kostant, Schmid [4, 7], this decomposition

is multiplicity free,

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m} \geq 0} \mathcal{P}_{\mathbf{m}}(V),$$

and the irreducible components can be parametrized by tuples $\mathbf{m} = (m_1, \dots, m_r)$ of integers satisfying $m_1 \geq \dots \geq m_r \geq 0$ (corresponding to certain highest weights). As an application of Theorem 2.2, we obtain a growth condition for polynomials in each component.

COROLLARY 2.4. *Let $z \in V$ be an element with singular values $\sigma_1 \geq \dots \geq \sigma_r \geq 0$. For any $p \in \mathcal{P}_{\mathbf{m}}(V)$ there exists $C > 0$ such that*

$$|p(z)| \leq C \cdot \sigma_1^{m_1} \dots \sigma_r^{m_r}.$$

Proof. Let (e_1, \dots, e_r) be a frame of tripotents. Recall from [10] that the following polynomial map is a highest weight vector of $\mathcal{P}_{\mathbf{m}}(V)$ (for an appropriate choice of a Borel subgroup of L),

$$p_{\mathbf{m}}(z) := \Delta_{\epsilon_1}(z)^{m_1-m_2} \cdot \Delta_{\epsilon_2}(z)^{m_2-m_3} \dots \Delta_{\epsilon_{r-1}}(z)^{m_{r-1}-m_r} \cdot \Delta_{\epsilon_r}(z)^{m_r}$$

where $\epsilon_k := e_1 + \dots + e_k$. For $p_{\mathbf{m}}$, Theorem 2.2 immediately yields

$$|p_{\mathbf{m}}(z)| \leq \sigma_1^{m_1} \dots \sigma_r^{m_r}.$$

For general $p \in \mathcal{P}_{\mathbf{m}}(V)$, irreducibility implies that there are $c_i \in \mathbb{C}$ and $k_i \in K$, such that

$$p(z) = \sum_{i=1}^s c_i \cdot p_{\mathbf{m}}(k_i z)$$

Since singular values are invariant under the action of K , this proves our statement with $C := \sum_i |c_i|$. \square

We refer to [8, Theorem 2.2] for an application of this growth condition. The main advantage of this estimate is the K -invariance of the right hand side.

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